

Are Logical Truths Analytic?

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ARE LOGICAL TRUTHS ANALYTIC?

THE TITLE of this paper may seem pointless. Nowadays the concept of analyticity is usually so characterized as to make all logical truths analytic by definition.¹ Hence, why the question?

The purpose of the title is not only to ask a question but also to challenge the ways in which the concept of the analytic is currently defined. This concept was brought into philosophical prominence by Kant;² I shall therefore examine some characterizations of this concept against the background of his use of it. It seems to me that the concept of analyticity as actually employed by philosophers like Kant is highly ambiguous and that most current definitions catch only one of the term's possible meanings.

I shall begin by listing a few of the ways in which the concept of the analytic (and, by implication, the concept of the synthetic) has been understood. The following explicit or implicit assertions have been made concerning analytic (analytically true) sentences:

I. They are true by sole virtue of the meanings of the terms they contain (analytic truth as conceptual truth).

II. They do not convey any factual information (analytic truth as tautological truth).

III. They can be shown to be true by strictly analytic methods.

The first interpretation of the concept of the analytic is often elaborated by remarking that truths of logic are as clear-cut examples of truths based solely on meanings as we are likely to have. This has inspired attempts to obtain all analytic truths by starting from the truths of logic and suitably extending their range. The following definition is probably the best-known attempt of this sort:

I (a). Analytically true sentences comprise the truths of logic

¹ One of the main sources here is Frege, who defined analytic sentences as those that can be proved by using only general logical laws and definitions. See *Foundations of Arithmetic*, trans. by J. L. Austin (Oxford, 1950), pp. 99 ff.

² As he was himself rather well aware of doing; he called his own distinction "klassisch" and "mächtig." See the Prolegomena, secs. 3 and 5 (pp. 270 and 276 in Vol. V of the Academy Edition). Cf. also Kant's Critique of Pure Reason, trans. by N. Kemp Smith (London, 1929), p. 55 (B 19).

together with all sentences reducible to them by substituting synonyms for synonyms.³

In this paper I shall disregard definitions of type I or I (a). Recent discussion has demonstrated, it seems to me, that they are unsatisfactory as they stand.⁴ Moreover, they make logical truths trivially analytic, and are therefore beside my present purpose.

But are there relevant senses of analyticity different from the one defined by I? I shall try to discover such senses by analyzing characterizations II and III. I shall first try to see somewhat more carefully what is implied by formulation III. Then I shall try to develop further characterization II so as to show that certain important truths of logic are analytic according to it. And finally I shall apply certain insights gained during the discussion to point out that the same truths are synthetic according to III, at least on one very natural interpretation of this characterization.

Let us therefore ask what can be said of sense III (that is, of the sense of analyticity defined by III). Here it is advisable to consider first the concept of an analytical argument-step instead of the concept of an analytic sentence. What can be said of the former can be subsequently extended to apply to the latter as well.

The basic idea of sense III seems to be expressible as follows:

III (a). All that is said by the conclusion of an analytic argument-step is already said in the premises.

This is admittedly very vague, largely owing to the vagueness of the notion of "saying" that is used here. For the purpose of definition III (a), this notion can be made somewhat clearer. In order to be able to speak of merely repeating or merely analyzing what is already said in the premises of an argument, we must restrict the sense of "saying" to what is in some sense actually or

³ Cf. W. V. O. Quine, From a Logical Point of View, 2nd ed. (Cambridge, Mass., 1957), pp. 22 ff.

⁴ Cf. Quine, *loc. cit.*; Morton G. White, *Toward Reunion in Philosophy* (Cambridge, Mass., 1955); and the evaluation of the subsequent discussion by Hilary Putnam in *Minnesota Studies in the Philosophy of Science*, III, ed. by Herbert Feigl and Grover Maxwell (Minneapolis, 1962), 359-360. I do not want to imply that every characterization of analyticity along the lines of I is beyond salvation, although I do think that characterization I(a) is seriously mistaken.

explicitly stated or mentioned in the premises. A traditional formulation of this idea was to say that the conclusion of an analytical argument-step merely repeats something already thought in the premises, although perhaps not yet with the same clarity and consciousness.⁵ Part of our task here is to see what objective explications such psychological or quasi-psychological formulations might have.

In spite of the vagueness which still remains in III (a), we can draw conclusions from it. The following criterion of analyticity will in any case follow from it:

III (b). In the conclusion of an analytic argument-step no more individuals are considered together at one and the same time than were already considered together in the premises.

For if more individuals are considered together in the conclusion than in the premises, some of them or some of their interrelations were not considered in the premises. Hence the conclusion does not consist in merely repeating what was already said in the premises, and the argument-step in question could not count as a case of "mere analysis."

Principle III (b) follows from III (a) no matter how the notion of "number of individuals considered together in a sentence" is to be understood. In the sequel I shall show how this notion can be clarified in the case of quantificational sentences.

A closely related consequence of criterion III (a) seems to be that the conclusion of an analytic argument cannot consider any individuals not already considered in the premises. Kant took this to imply the following principle:

⁵ Cf. Kant on his distinction between the analytic and the synthetic. In an analytic judgment "I have merely to analyse the concept, that is, to become conscious to myself of the manifold which I always think in that concept" (Kemp Smith, *op. cit.*, p. 49). In a synthetic judgment we sometimes "are required to join in thought a certain predicate to a given concept, and this necessity is inherent in the concepts themselves. But the question is not what we *ought* to join in thought to the given concept, but what we *actually* think in it, even if only obscurely" (*ibid.*, pp. 53-54; the italics are Kant's). The last sentence is remarkable in that Kant there explicitly countenances sentences which turn on "necessities inherent in the concepts themselves," that is, which are analytic in sense I, but which for Kant are nevertheless synthetic. This suggests that Kant's intentions are not very well served by an explication of analyticity along the lines of sense I.

III (c). An analytic argument never carries us from the existence of an object to the existence of a different object.

In short, according to Kant interindividual inferences concerning existence are impossible by analytic means.⁶

It is not difficult to see how similar considerations might be applied to longer arguments, and hence to sentences established by means of such arguments. There is even more than one way of doing so. We might call a proof of q from p analytic if all its steps are analytic in one of the senses just indicated. This approach does not appear to be as interesting, however, as a slightly different one in which the proof in question is considered not only from the point of view of the premise p but also from that of the conclusion q. Then a proof of q from p is analytic in sense III (b) if no more individuals are considered at any of the intermediate stages than are already considered either in p or in q. This is the sense of analyticity as applied to proofs which I shall be using in what follows. A logically true sentence p of quantification theory may then be called analytic if it can be proved analytically, in the sense just explained, from a propositional tautology in which no more individuals are considered together than in p. A sentence will be called analytically inconsistent if a propositional contradiction in which no more individuals are considered together can be derived from it analytically.

I shall return to these senses of analyticity later. Meanwhile, I shall stage my first main attack on analyticity in the direction of sense II. About forty years ago, a notion very much like this one was prominent. This was the notion of tautology of Wittgenstein's *Tractatus*. Unfortunately, the original form of this notion was satisfactorily defined only for propositional logic. Certain im-

⁶ This view of Kant's is a generalization of his view on what he called "Hume's problem." For an early formulation of the problem, see the Academy Edition of Kant's works, II, 202-203. The general problem is there formulated as follows: "Wie soll ich es verstehen, dass weil Etwas ist, etwas Anderes sei?" Kant's answer is that this cannot happen "durch den Satz des Widerspruchs," that is, analytically. Similar formulations occur in the first Critique, in the Prolegomena, and even in the Critique of Practical Reason; cf., e.g., Prolegomena, Academy Edition, IV, 257, 260, 277-278; the last passage shows the intimate connection between the justification of interindividual existential inferences and Kant's main problem of justifying synthetic judgments a priori.

portant generalizations have been suggested since, but for one reason or another they seem to have less philosophical interest than Wittgenstein's original notion.

I want to argue, nevertheless, that something like Wittgenstein's notion of tautology can be generalized in a natural and informative way so as to be applicable in quantification theory. In order to see what the generalization is, we have to see what makes his original notion so appealing.

It is made so, I think, by the fact that in propositional logic one can actually list all the "possible worlds" that we can describe by means of a given supply of atomic sentences.

If we are given the atomic sentences p_i (i = 1, 2, ..., k), the descriptions of the possible worlds are conjunctions which for each *i* contain either p_i or its negation $\sim p_i$ (but not both) but which do not contain any other members. Following a time-honored precedent established by Boole, I shall call these conjunctions the *constituents* of propositional logic, and I shall designate

an arbitrary constituent by
$$\prod_{i=1}^{i=k} p_i$$
, or, more simply, by $\prod_{i=1} p_i$.

Different constituents may be distinguished from each other by attaching subscripts to Π . These subscripts are assumed to run consecutively from one onward, so that the same notation can be applied repeatedly.

An arbitrary constituent $\prod_{i=1}^{j} p_i$ may also be said to be of the

form

 $(\pm) p_1 \& (\pm) p_2 \& \dots \& (\pm) p_k.$

Here each symbol (\pm) stands either for a negation sign or for nothing at all. For different patterns of negation signs a different subscript j is chosen.

Why does the existence of the constituents make the notion of tautology appealing? Because each consistent sentence of propositional logic has a normal form which is a disjunction of some (perhaps all) of the constituents. In an obvious sense, every sentence considered in propositional logic thus admits some of the possibilities listed by the constituents, but excludes the rest. In an equally obvious sense, the more possibilities it excludes, the more informative it is.⁷

A limiting case is that of a sentence admitting all the possibilities listed by the constituents, but excluding none of them. Such a sentence is empty in a very obvious sense of the word: it cannot convey any genuine information. And this limiting case is just that of the logically true sentences of propositional logic. They are undoubtedly true, but in the striking sense just explained they do not carry any information concerning the subject matter of which they apparently speak.

These are the facts, it seems to me, that make Wittgenstein's notion of tautology so very appealing.

They immediately suggest a more general sense in which we may ask whether the truths of other parts of logic are also tautologous. This sense is not quite sharply defined yet, but we can nevertheless understand what is being asked. In any part of logic we may ask: is it always possible to list all the alternatives concerning the world in such a way that the truths of this part of logic are just the sentences admitting all these alternatives, but excluding none? If so, the truths of this part of logic are so many tautologies.⁸

This connection between the exclusion of possibilities and the amount of information a sentence conveys was first made explicit by Karl Popper and has subsequently been emphasized by him; see, e.g., *The Logic of Scientific Discovery* (London, 1959), esp. secs. 23, 31, and appendix *ix.

On the basis of this idea one can readily construct some very natural measures of the information which a sentence carries in propositional logic and in monadic quantification theory. Cf. Yehosua Bar-Hillel and Rudolf Carnap, "Semantic Information," *The British Journal of the Philosophy of Science*, IV (1953-1954), 147-157. A tautology may then be defined simply as a sentence with zero information. In propositional logic it turns out that the tautologies so defined coincide with the logical truths of propositional logic.

⁸ This question may be reformulated as a question whether one can extend in a natural and informative way the simple measures of semantic information which can be defined in propositional logic to other parts of logic in such a

⁷ Suppose I know a sentence to be true. Then I can, e.g., leave out of consideration in making my plans and decisions all the alternatives excluded by the sentence. Clearly, the more possibilities I can thus rule out, the more I can say that I know. If I cannot rule out any possibilities, then I know nothing at all about the subject matter at hand. As Wittgenstein says, "I know nothing about the weather if I know that it is raining or not raining" (*Tractatus* 4.461).

Let us study quantification theory as a test case. Can we list all the possibilities concerning the world that can be expressed by means of the resources employed in some given quantificational sentence?

The answer depends on the meaning we assign to the expression "by means of the resources employed in some given sentence." If this is taken to mean "by means of the predicates occurring in the given sentence" (plus quantifiers and propositional connectives, of course), then in most cases there is no hope of making a finite list of the desired kind.⁹

But if we introduce further limitations, the answer is different. For each sentence which is considered in quantification theory, there is a maximum to the lengths of the sequences of nested quantifiers occurring therein. More popularly expressed, each quantificational sentence is characterized by the number of the layers of quantifiers it contains. This number will be called the *depth* of the sentence in question. In other words, the depth of a sentence is the maximum number of quantifiers whose scopes all overlap in it.¹⁰ Each sentence has, moreover, a finite number of

way that a sentence is logically true if and only if the information it carries is zero. This question has not been answered so far even in the case of the whole of quantification theory. The measures of semantic information proposed by Bar-Hillel and Carnap (*op. cit.*) yield unnatural results if one tries to extend them to the whole of quantification theory. According to these measures, every existentially quantified sentence carries zero information in an infinite domain of individuals, even if it is not logically true. If the domain of individuals is not allowed to become infinite, these measures assign a zero information to every sentence that is not logically true but whose negation is satisfiable only in an infinite domain. (The denial of an axiom of infinity would be a case in point.)

⁹ For if we could have such a finite list, we would have a decision method for many cases in which it is known to be unavailable.

¹⁰ The depth d(p) of an arbitrary quantificational sentence p may also be defined recursively as follows: d(q) = 0 if q is an atomic sentence or an identity; $d(q_1 \& q_1) = d(q_1 \lor q_2) = max(d(q_1), d(q_2)) =$ the greater of the two numbers $d(q_1), d(q_2)$; d((Ex)q) = d((x)q) = d(q) + 1 (if q is here not a sentence and if d(q) is therefore undefined, we may use instead the depth of any sentence obtained from q by substituting a free individual symbol for x). For instance, we have d(P(a, b)) = 0; d((Ey)P(a, y)) = d(P(a, b)) + 1 = 1 = d((Ey)P(y, a)); d((Ey)P(a, y) & (Ey)P(y, a)) = max(d((Ey)P(a, y), d((Ey)P(y, a))) = max(1, 1) = 1; and hence d(x)((Ey)P(x, y) & (Ey)P(y, x))) = 2. Similarly, the depth of $(x)((Ey)P(x, y) \lor (Ey)(Ez)(P(y, z) \& P(z, x)))$ is 3.

free individual symbols (constants or free variables). The sum of this number and the depth of the sentence in question will be called its *degree*. If we now consider what can be expressed by means of sentences constructed from a given finite supply of predicates and free individual symbols plus quantifiers and having a degree smaller than a given positive integer, there is a way of listing all the different alternatives concerning our universe of discourse that can be expressed by means of these resources.

The limitation on the degree of the sentences is a natural one, for the notion of the degree of a sentence has a very simple intuitive meaning. The degree of a sentence is the maximum number of individuals we are considering at any one time in their relation to each other in the sentence.

Since this intuitive meaning of the notion of degree will be important in what follows, it is worth explaining carefully. For this purpose, we may ask: how are individuals introduced into our arguments? Part of an answer is obvious: individuals are introduced into our reasoning by free individual symbols. This gives us the first of the two addenda whose sum is the degree of a sentence. This answer is only a partial one, however, for individuals are introduced into our propositions also by quantifiers (bound individual variables). In order to see this, it suffices to recall the most accurate translations of quantifiers into more or less ordinary language: "(Ex)" is to be read "there is at least one individual (let us call it x) such that" and "(x)" is to be read "for each individual (call it x) it is the case that." Each quantifier thus invites us to consider exactly one new individual, however indefinite this individual may be. Two quantifiers whose scopes do not overlap cannot both be counted here, however, for there is no way of relating to each other the individuals which such quantifiers invite us to consider. Hence the contribution of quantifiers to the maximal number of individuals we are considering together in a certain sentence is the maximal number of quantifiers whose scopes overlap in it, exactly as was suggested.¹¹

¹¹ But are the individuals which nested quantifiers invite us to consider necessarily *different* individuals, as we seem to have assumed in counting them all? (This question was first raised to me by Professor Hector-Neri Castañeda.) The answer is simple, and instructive. The individuals which nested quantifiers

Another way of seeing the intuitive meaning of the notion of a degree in quantification theory is to ask the question: what are the individuals whose properties and interrelations you are considering (or may consider) in a given part of a quantificational sentence, say between a certain pair of parentheses? Obviously, they include the individuals referred to by the free individual symbols of the sentence. They also include all the indefinite individuals introduced by the quantifiers within the scope of which we are moving. They do not include any other individuals. The maximum number of these individuals is just the degree of the sentence in question, which is therefore the maximum number of individuals we are considering together in the sentence.

This informal explanation has a neat formal counterpart. If it is required, as is natural, that quantifiers with overlapping scopes must have different variables bound to them, then the depth of a sentence is the least number of different bound variables one needs in order to write it out, and its degree is therefore the least number of different individual symbols (free or bound) one needs in it.

introduce into our reasoning are necessarily different if and only if quantifiers are given what I have called an exclusive interpretation. (For this interpretation, see my article, "Identity, Variables, and Impredicative Definitions," Journal of Symbolic Logic, XXI [1956], 225-245.) Indeed, the difference between the usual "inclusive" and the new "exclusive" interpretation of quantifiers lies in this very requirement. Hence we must, strictly speaking, apply our notion of a degree only to quantificational sentences with exclusively interpreted quantifiers. It is not very difficult, however, to translate sentences with inclusively interpreted quantifiers into a language which makes use of exclusively interpreted quantifiers only (see *ibid*.). The degree of the translation may then serve as the degree proper of the original (inclusively interpreted) sentence. It turns out, moreover, that this translation very rarely makes any difference to the degree of our quantificational sentences. For this reason, the requirement of an exclusive interpretation of quantifiers makes little difference here. In fact, not very many points made in this paper turn on the peculiarities of an inclusive interpretation, and those that turn on it can easily be rewritten in terms of an exclusive interpretation. Cf. also Hintikka, "Distributive Normal Forms in First-Order Logic," in Formal Systems and Recursive Functions, ed. by John N. Crossley (to be published).

Similar remarks pertain of course also to the question whether the values of bound variables may coincide with the referents of free individual symbols occurring in the same sentence. Here, too, a change in the interpretation of quantifiers is called for in order to make my definition of degree applicable. Again, the change is so small as to make no difference to my present purposes, however. The intuitive meaning of the degree of a sentence is straightforward enough to have already caught the eye of C. S. Peirce, at least in a simple special case. (See his *Collected Papers*, 3.392 : "The algebra of Boole affords a language by which anything may be expressed which can be said without speaking of more than one individual at a time.")

But does not a general sentence speak of all the individuals of the domain (universe of discourse)? Is not the number of individuals considered in such a sentence therefore infinite if the domain is infinite? Surely a general sentence does speak in some sense of all the individuals in the domain; but in such a sentence we are not considering all these individuals in their relation to each other. In a sentence like "All men admire Buddha" we are not considering the interrelations that obtain between any two men. We are, so to speak, considering each man at a time and saving something about his relation to the great Gautama. Hence the number of individuals considered in their relation to each other in this sentence is two, which is just its degree. In the first half of the sentence "John has at least one brother and John has at least one sister" we are considering John in his relation to an arbitrarily chosen brother of his, and in the second half we are considering him in his relation to one of his sisters. Nothing is said, however, of the relations between his brothers and his sisters. Hence the number of individuals considered together at any given time in the sentence is only two, which is again exactly its degree. This illustrates the fact that quantifiers with nonoverlapping scopes do not count in the total. By contrast, in the sentence "All John's sisters are older than his brothers" an arbitrary brother of John's is compared as to age with an arbitrary sister of John's; hence the number of individuals considered in their relations to each other is three, again equaling the degree of the sentence.

These examples illustrate the intuitive meaning of our notion of degree. Apart from this intuitive meaning, it plays an interesting role in quantification theory. If a limit is imposed on the degrees of our sentences, we have in quantification theory a situation strongly reminiscent of propositional logic. Given a finite supply of predicates and free individual symbols, there is a finite number of constituents such that every consistent sentence considered has a normal form in which it is a disjunction of some (perhaps all) of the constituents. I shall not prove this result here. I have done so in a number of other papers in which I have also considered these "distributive normal forms" in certain other respects.¹²

It is not my purpose here to examine the structure of distributive normal forms in any greater detail. There are two questions concerning them which must nevertheless be discussed. First, we want to make sure that the constituents occurring in them really list all the alternatives concerning the world in as clear-cut a sense as do the constituents of propositional logic. Secondly, we have to ask whether the logical truths of quantification theory are related to the constituents in the same way as are the logical truths of propositional logic.

I think that the first point can be sufficiently established by considering, by way of example, some of the simplest kinds of quantificational constituents. If there are no free individual symbols present and if we have merely a number of monadic (one-place) predicates $P_i(x)$ (i = 1, 2, ..., m), the constituents will have the following form:

(I)
$$\prod_{k=1}^{k=2^m} (Ex) \prod_{i=1}^{i=m} P_i(x)$$

This is in a clear-cut sense a description of one kind of a "possible world." It is easy to see how this description is accomplished. First, we list all the possible kinds of individuals that can be specified by means of the predicates $P_i(x)$. This is what the conjunctions

(2)
$$\prod_{i=1} P_i(x) = C_k(x)$$

 $(k = 1, 2, \ldots, 2^m)$ do. Then we specify, for each such kind of individuals, whether individuals of that kind exist or not. It is perhaps not entirely surprising that everything we can say by using only the predicates $P_i(x)$, quantifiers, and propositional

¹² See Hintikka, op. cit.; "Distributive Normal Forms in the Calculus of Predicates," Acta Philosophica Fennica, VI (1953); "Distributive Normal Forms and Deductive Interpolation," Zeitschrift für mathematische Logik und Grundlagen der Mathematik, X (1964), 185-191.

connectives is a disjunction of such descriptions of kinds of possible worlds.

A simple example perhaps makes the situation easier to appreciate. Suppose that m = 2, that is to say, suppose that we are given two monadic predicates, say "red" and "round." Then conjunctions (2) are of the form

$$(\pm)$$
 (x is red) & (\pm) (x is round).

They specify all the different kinds of individuals that can be specified by means of the two predicates:

Each constituent of form (1) indicates, for each of the different kinds of individuals $(2)^*$, whether such individuals exist or not. To take a random example of constituents of form (1), one of them will be the following sentence:

There are individuals which are red and round; there are no individuals which are red but not round; there are no individuals which are round but not red; there are individuals which are neither red nor round.

We can also see an interesting way of rewriting a constituent of form (1). Instead of listing all the different kinds of individuals that exist and also listing the kinds of individuals that do not exist, it suffices to list the kinds of existing individuals and simply to add that they are all the kinds of individuals in existence. This means that each constituent of form (1) can be rewritten so as to be of form

(3)
$$(Ex)C_1(x) \& (Ex)C_2(x) \& \dots \& (Ex)C_n(x) \&$$

(x) $(C_1(x) \lor C_2(x) \lor \dots \lor C_n(x)),$

where $\{C_i(x)\}$ (i = 1, 2, ..., n) is some subset of the set of all conjunctions (2). It can be shown that all the constituents of quantification theory may be similarly rewritten.

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For instance, the constituent which was formulated in words above could obviously be rewritten as follows:

There are individuals which are red and round as well as individuals which are neither red nor round; and every individual is either red and round or neither red nor round.

In order to have more insight into the structure of our constituents, let us assume that we are given a number of dyadic (two-place) predicates $R_i(x, y)$ (i = 1, 2, ..., r) but no other predicates nor any free individual symbols, and that the depth of our sentences is at most two. Then constituents are still of form (3). In fact, (3) may be said to be the general form of those constituents which do not contain free individual symbols. The definition of the conjunctions $C_i(x)$ has to be changed, however, from case to case. In the case at hand, each $C_i(x)$ is rather like (3):

(4)
$$(Ey)C'_{1}(x, y) \& (Ey)C'_{2}(x, y) \& \dots \& (Ey)C'_{s}(x, y)$$

 $\& (y) (C'_{1}(x, y) \lor C'_{2}(x, y) \lor \dots \lor C'_{s}(x, y)) \&$
 $\prod_{i=1}^{i=r} R_{i}(x, x)$

Here each C'(x, y) is of the form

$$\prod_{i=1}^{i=r} R_i(x,y) \& \prod_{i=1}^{i=r} R_i(y,x) \& \prod_{i=1}^{i=r} R_i(y,y).$$

The intuitive meaning of (4) is not very difficult to fathom. In effect, we first list all the different ways in which an individual ymay be related to a given individual x. Given a fixed x, this list is also a list of different kinds of individuals y (in their relation to x). Then we specify which of these different kinds of y exist for some fixed x. (We specify, furthermore, the ways in which x is or is not related to itself.) What I am saying is that this gives us a list of all the possible kinds of individuals x that we can specify by using only the dyadic predicates $R_i(x, y)$, quantifiers, and propositional connectives, provided we do not make use of sentences of a degree higher than one.

What happens in (4) may also be described as follows. We took a list of all the relations (two-place predicates) which may obtain between two individuals and which can be specified without using quantifiers, and we constructed out of them a list of all the possible complex attributes (one-place predicates) which an individual may have and which may be specified by means of just one layer of quantifiers. It is a straightforward task to generalize this: in the same way we may start from the list of all the possible relations which may obtain between n + 1 individuals and which can be specified by m layers of quantifiers, and construct out of them a list of all the different relations which can obtain between n individuals and which can be described by means of m + 1 layers of quantifiers. In this way we may in fact easily obtain an inductive definition of constituents in general, for in the case m = 0 we have simply constituents in the sense of propositional logic. (Such a definition is of course relative to a given finite list of predicates and free individual symbols.)

These examples and indications perhaps suffice to show that the constituents of quantification theory really give us a systematic list of all the different possibilities concerning reality which can be specified by the means of expression that we have at our disposal, in the same sense in which the constituents of propositional logic do so. We could also use these constituents in the same way as the constituents of propositional logic have sometimes been used, namely to develop measures of the information which a sentence carries. Tautologies would then be sentences with zero information. In other respects, too, the situation is exactly the same in quantification theory as it is in propositional logic, with but one important exception. This one difference between the two cases is that in quantification theory some constituents are inconsistent whereas no constituents of propositional logic are.¹³

The question we have to ask is whether this makes the situation essentially different from what it is in propositional logic. It may

¹³ More accurately, this is what distinguishes constituents of degree two or more from those of degree one: the former may be inconsistent, but the latter never are. The interesting special case of monadic quantification theory reduces to the second of these two types of cases. Hence the situation in monadic quantification theory is exactly the same as in propositional logic; and hence the truths of the former are tautologies in exactly the same sense as those of the latter. This already shows that our notion of tautology has interesting applications outside propositional logic.

appear that it does make a difference. The fact that there are inconsistent constituents implies that a sentence may be logically true even though its distributive normal form does not contain all the constituents, provided that the missing constituents are all inconsistent. Thus it may appear that the truths of quantification theory need not be tautologies in the sense of admitting all the alternatives that we can specify with respect to the world. It suffices for them to admit all the alternatives specified by the consistent constituents.

An answer lies close at hand. We suggested defining a tautology as a sentence which admits of all the possibilities that there are with respect to the world. Now it is perfectly natural to say that an inconsistent constituent does not specify a genuine possibility concerning the subject matter it seems to be speaking of, but only appears to describe one. Just because it is inconsistent, the state of affairs it purports to describe can never be realized, so there is no need for any sentence to exclude it. Hence a necessary and sufficient condition for a sentence of quantification theory to admit all the kinds of worlds which are really possible that is to say, to be a tautology—is that its distributive normal form contain all the *consistent* constituents. And it is readily seen that all the truths of quantification theory really are tautologies in this sense.

This way out of the difficulty may seem far too simple. It can be strengthened, however, by means of further arguments.

I shall here give in the form of an analogy an argument which, although merely persuasive, can be converted into a stricter one. If we are given a constituent, we are not yet given a genuine picture of a possible state of affairs. We are given, rather, a way of constructing such a picture—as if we were given a jigsaw puzzle. In fact, (3) shows that being given a constituent is very much like being given an unlimited supply of a finite number of different kinds of pieces of a jigsaw puzzle, with two instructions: (i) at least one piece of each kind has to be used; (ii) no other kinds of pieces may be used. An attempt to construct "a picture of a possible state of affairs" in accordance with these instructions may fail. Then the jigsaw puzzle does not give any picture of reality: it cannot be used to convey information concerning the state of the world. We cannot give it to somebody and say "This is what the world is like" and hope to convey any real information to him as we could have done by giving him a ready-made picture of the world or even a jigsaw puzzle which might yield a genuine picture. Similarly, it may be suggested, an inconsistent constituent does not describe a genuine possibility as to what the world may be like but only appears to do so. Hence its presence or absence makes no difference to the normal forms: no knowledge of the subject matter of which the sentence in question speaks is needed to rule it out.

This analogy can be made stronger in two ways. It may be argued that what most directly specifies the structure of the world (and in this sense gives us the "real meaning" of a constituent) is not the constituent itself but rather the outcome of those operations that we have compared to the construction of a jigsaw puzzle. Such an argument might take the form of a defense of a rudimentary form of what is known as the picture theory of language. On this view, a constituent or a sentence of some other kind is not itself a "picture" of a possible state of affairs, but rather gives us a starting point for the construction of ap icture or a set of alternative pictures. It would ta keus too far afield, however, to develop this idea here.

It will have to suffice to give a single reason for the aptness of the jigsaw puzzle analogy. This is the fact that it very well reproduces the reasons why inconsistent constituents are inconsistent. In order to see two such reasons, we may consider sentences (3) and (4) and assume that the latter occurs as a part of the former.¹⁴ Both (3) and (4) are lists of all the kinds of individuals that there are. In the first list these individuals are classified absolutely, in the second with respect to the given

¹⁴ These two reasons for the inconsistency of certain constituents are explained in a more systematic way in "Distributive Normal Forms in First-Order Logic" (see note 11) as conditions (A) and (B). Strictly speaking, a third condition (C) is also needed. This condition is relatively trivial, however, as is shown, e.g., by the fact that it can be dispensed with if we use an exclusive interpretation of quantifiers. Hence I shall disregard it here. The conditions (A) and (B) have been discussed in an instructive special case by G. H. von Wright under the suggestive names "fitting-in problem" and "completion problem," respectively. See his *Logical Studies* (London, 1957), p. 50.

individual x. Nevertheless, the two lists have to be compatible for every $C_i(x)$ —that is, for each sentence of the form (4) which occurs in (3). For, clearly, every individual that exists according to the absolute list has to find a place in the relative list of each existing individual, and vice versa. These two requirements are not always met. If they are not, (3) is inconsistent. If the first requirement is violated, (3) may be shown to be inconsistent by essentially one application of the exchange theorem (Ex)(y)P(x, y) $\supset(x)(Ey)P(y, x)$. If the second is violated, (3) may similarly be shown to be inconsistent by essentially one application of the exchange theorem $(Ex)(Ey)P(x, y) \supset (Ex)(Ey)P(y, x)$.

If I am right, these two are essentially the only ways in which a constituent can turn out to be inconsistent. Of course, this cannot mean that every constituent which is not inconsistent for one of these two reasons is thereby shown to be consistent. Often the failure of a constituent to meet the two requirements is implicit and becomes explicit only when the constituent in question is expanded into a disjunction of several constituents of a greater depth. At some finite depth, each of these deeper constituents will then be inconsistent for one of our two reasons.¹⁵

Here the jigsaw puzzle analogy serves us remarkably well. I may sum up my explanation of the two reasons why a constituent may be inconsistent by comparing one of my inconsistent constituents to a jigsaw puzzle which can fail to yield a coherent picture, for two reasons. Either there are two pieces (or, rather, kinds of pieces) which are incompatible in the sense that they cannot be fitted into one and the same picture; or else one of the pieces leaves a gap which is such that it cannot be filled by any of the different kinds of pieces that are at our disposal. The former case arises when some member of the absolute list cannot find a niche in the relative list of one of its fellow members; then the two

¹⁵ This follows from the completeness theorem of "Distributive Normal Forms in First-Order Logic." Every provable formula of quantification theory thus has in principle a proof of an especially simple structure. In each branch of the proof, only such relatively trivial rules are needed as enable us to convert formulae into distributive normal form and to add redundant parts to them so as to increase their depth, with but one essential exception in each branch. This exception is an application of one of the rules for changing the order of adjacent quantifiers.

members of the absolute list are incompatible. The latter case arises when some member of the relative list of some fixed $C_i(x)$ does not fit into the absolute list; then this $C_i(x)$ "leaves a gap" which cannot be completed by any of the members of the absolute list of which we are allowed to make use. The fact that we sometimes have to expand the given constituent into a disjunction of several constituents of a greater depth may be compared to the fact that we sometimes have to carry an attempted construction of a jigsaw puzzle to a certain extent before it can be seen that it is impossible to complete for one of the two reasons which I just mentioned.

This success of the jigsaw puzzle analogy will reinforce the point which I made by its means: knowing that certain constituents are inconsistent does not give us any information concerning the reality which the constituents purport to speak of and hence does not interfere with the tautologicality of the logical truths of quantification theory.

Our observations thus strongly suggest that the truths of quantification theory are really analytic in sense II—that is, tautologies in the sense in which we have decided to use the term.

In the course of our discussion, we have already found indications that some of the logical truths of quantification theory are nevertheless *not* analytic in our sense III—that is, not provable by analytic methods. Now I shall argue more fully for this second main point.

For those truths of quantification theory that do not turn on the elimination of any constituents, it may be argued that they are analytic in sense III.¹⁶ But for the rest the situation is entirely different. The briefest glimpse already suggests that the incon-

¹⁶ If so, all the truths of monadic quantification theory (monadic predicate logic) are analytic also in sense III. For (as mentioned in note 13) the truths of this part of logic do not depend on the elimination of inconsistent constituents.

This observation may be of some historical interest. In logic, the attention of most traditional philosophers, Kant included, was confined to traditional syllogistic, which is a part of monadic quantification theory. The fact that senses II and III of analyticity coincide with each other and with logical truth in this area may have been instrumental in leading traditional philosophers to think that they coincide everywhere.

sistency of some of the constituents is essentially connected with sense III of the analytic and the synthetic. If a constituent like (3) is inconsistent, then the following implication is provable.¹⁷

(5)
$$((Ex)C_1(x) \& (Ex)C_2(x) \& \dots \& (Ex)C_n(x)) \supset (Ex)(\sim C_1(x) \& \sim C_2(x) \& \dots \& \sim C_n(x)).$$

This is clearly an instance of the kind of interindividual existential inference which for Kant constituted the paradigm of synthetic inferences (cf. criterion III[c] formulated above). In this case the difference between the different individuals that Kant speaks of is understood in the strongest possible sense, to wit, in the sense of logically necessary difference. Conversely, it may be argued (with certain qualifications) that in every logically valid inference from the existence of a number of individuals to the existence of another individual which is for logical reasons different from them there is implicit the inconsistency of at least one of our constituents. In short, inconsistencies of constituents would have been for Kant paradigmatic instances of synthetic truths of (modern) logic.

There are other ways of arguing that the elimination of the inconsistent constituents is a synthetic procedure in sense III of the analytic and the synthetic. An especially clear-cut one is given by criterion III(b). It was suggested earlier that one way of showing that a constituent is inconsistent is to transform it into a disjunction of constituents of a greater depth and therefore of a higher degree and to show that all of these are inconsistent for one of the two reasons I explained. Now the intuitive meaning of the notion of the degree of a sentence is, as I indicated, that of the maximum number of individuals that we are considering together at one and the same time in the sentence in question. According to criterion III(b), this number must not be greater in any of the sentences by means of which a given sentence p is proved or disproved than it is in p itself, if this proof or disproved

¹⁷ It is important to realize that the provability of this implication does not usually turn on the contradictoriness of its antecedents and that it therefore is not normally of the trivial kind. In fact, the antecedents of (5) are typically satisfiable. If one of them is not satisfiable, then this merely means that there occurs as a part of (3) an inconsistent constituent of lesser depth which gives rise to the nontrivial provability of an implication of the same form (5).

is to be analytic. Since the procedure I just mentioned for eliminating an inconsistent constituent makes use of sentences of a degree higher than that of the constituent to be eliminated, it is a synthetic procedure in the sense of criterion III(b).

Is this perhaps an accidental peculiarity of my procedure? I do not think so; on the contrary, I think it an unavoidable feature of every complete proof procedure in quantification theory, in some fairly natural sense of "proof procedure." Every such proof procedure must make frequent use of sentences of higher degree than that of the sentence to be proved. This is made inevitable by the fact, noticed earlier, that only a finite number of nonequivalent sentences can be made by means of the predicates and free individual symbols occurring in a given sentence, if a limitation is imposed on the degree of these sentences. If our rules of inference do not affect this degree, they cannot lead us out of this finite set of sentences. If certain fairly natural limitations are imposed on these rules of inference, it will be possible to show that this would give us a decision procedure, which is known to be nonexistent in many cases. In order to give us a complete proof procedure, our rules of inference must therefore allow proofs of sentences by means of sentences of higher degrees. Such a proof procedure will then be synthetic in our sense III(b).

The limitations that have to be imposed on rules of inference in order for what was just said to be true have some interest in themselves. Sometimes a rule of inference in the most general sense of the word is essentially identified with a two-place recursive predicate of the Gödel numbers of sentences (or formulae, if you prefer). In this wide sense, we can indeed have rules of inference which are analytic in sense III(b) and which nevertheless enable us to prove all (and only all) the logical truths of quantification theory.¹⁸ I think this sense in any case far too broad, however, to constitute a natural explication of what we would naturally mean

¹⁸ For instance, the following "rule of proof" might be used (in combination with suitable propositional rules): From $(p \& p \& \ldots \& p)$ (a conjunction with k members) infer q if and only if k is the Gödel number of a proof of q from p (in some standard formulation of quantification theory). This rule obviously suffices for all the proofs we want; and its applicability to a pair of formulae is obviously a recursive predicate of the Gödel numbers of these formulae. (A rule of this kind was first suggested to me by William Craig.)

by a rule of inference. We have to require that the applicability of such a rule depends, intuitively speaking, only on what the sentences in question express or say and not on accidental features of their formulation. This requirement may seem too vague to be useful; nevertheless it has some very definite implications. For instance, it may be taken to imply that a rule of inference must be independent of the way in which truth functions are written out; and it must be independent of the particular free individual symbols which occur in the sentences in question. Hence only such two-place recursive predicates of Gödel numbers of sentences will qualify for a rule of inference as are invariant with respect to arbitrary replacements of truth functions by tautologously equivalent truth functions (of the same arguments) in the sentences in question (or which can be extended so as to become invariant in this sense without affecting provability relations).¹⁹ Such replacements must of course be admissible also inside larger sentences. Moreover, we must require symmetry with respect to the different free individual symbols.

If these natural restrictions are imposed on what we are willing to call a rule of inference, it may be shown that no set of rules of inference analytic in sense III(b) suffices to enable us to prove each logical truth of quantification theory from propositional tautologies of the same degree or to carry out analytically (in sense III[b]) all the proofs from premises which we would like to carry out.²⁰

It is easy to verify that most of the familiar proof procedures in quantification theory satisfy the two requirements, and also that

¹⁹ The rule mentioned in note 18 does not satisfy this requirement because its applicability depends on the number of identical members of a conjunction which of course does not make any difference to it as a truth function.

²⁰ It is easy to see that the restrictions which I mentioned enable us to define a normal form of quantificational sentences which is somewhat cruder than the distributive normal form but which has the following properties. (a) If the free individual symbols and predicates of our sentences are limited to a finite set and if an upper bound is imposed on their depth (degree), then there is only a finite number of normal forms. (b) A sentence p can be inferred from q if and only if the normal form of the former can be inferred from the latter. Then it is easy to see that if we had a complete rule of inference which does not affect the degrees of our sentences, we should have a decision method for the whole of quantification theory, which is known to be impossible.

they in fact allow proofs of sentences by means of higher-degree sentences, that is to say, proofs synthetic in the sense we are now considering. An innocent exception is constituted by some of the natural deduction methods, where the process known as existential instantiation does not depend solely on the existential quantified sentence to be instantiated.²¹ Usually it has to be required that the instantiating free individual symbol is different from all the free individual symbols occurring earlier in the proof. This means that existential instantiation is not independent of the particular free singular terms occurring in the result of the process of existential instantiation. In deciding how many individuals are considered together in a step of a proof by natural deduction methods, we therefore have to count not only the free individual symbols which occur in the premises of this particular step but also all the ones that occur at earlier stages of the proof. We may, for example, consider the conjunction of all the sentences reached up to a certain point and consider its degree instead of the degree of the individual lines of proof. (It may be called the degree of the set of sentences so far reached.) But if we do this, we find that natural deduction methods also conform to the pattern we have found. In them, too, we frequently have to add to the number of individuals we are considering together in order to be able to carry out the proofs we want to carry out; in other words, we have to add to the degree of the sets of sentences we are considering.

Natural deduction methods are interesting from our point of view because the synthetic element in them may be reduced to a single rule. In a suitable formulation of these methods there is only one rule that is synthetic, that is, that adds to the number of individuals one is considering in the sense just explained. This is

²¹ Existential instantiation (specification, exemplification) is the transition from an existentially quantified sentence (Ex)p to a sentence instantiating it, e.g., p(a/x), where a is a free individual symbol and p(a/x) the result of replacing x by a in p (wherever it is bound to the initial quantifier of [Ex]p). The rule which allows us to make this transition is not symmetrical with respect to the different free individual symbols, however. Those free individual symbols have to be barred from the role of a which occur at the earlier stages of the proof, even if they do not occur in the premise (Ex)p itself. This restriction makes the natural deduction methods slightly unnatural from our point of view.

just the rule of existential instantiation. If the other rules are formulated in a suitable way, the rule of existential instantiation thus takes all the blame for increasing the degree of the sets of sentences we have to consider in order to establish a logical truth of quantification theory. (Universal instantiation need not increase the degree, for it may be restricted to the cases where the instantiating free individual symbol is an old one, that is, occurs earlier in the argument.)

I conclude, then, that the logical truths of quantification theory cannot all be captured by (natural) rules of inference which would be analytic. In this sense, quantification theory is a synthetic theory.

We might also try to spell out more clearly which particular logical truths of quantification theory are synthetic in the most natural sense of the word based on criterion III(b). Such an attempt would bring us back, it seems to me, to the distinction between logical truths depending on the elimination of inconsistent constituents and those not depending on it. Almost all the logical truths as well as almost all the usual logical arguments that one finds in ordinary textbooks of logic will then turn out to be analytic in the relevant sense of the term, the main exception probably being offered by the laws for exchanging adjacent quantifiers. The details need not detain us here, however, since the main point is clear enough. The fact that many logical truths of quantification theory turn out to be analytic in one sense of the word but synthetic in another sense shows the importance of the distinction between the different senses. It also shows that it does make sense to ask whether the logical truths of quantification theory are analytic.

It remains to make good my promise to relate our findings to Kant's distinction between the analytic and the synthetic. I shall confine myself to a few general remarks only.

The basis of the connection may be expressed in terms of the history of the notions of the analytic and the synthetic. Like so many other important philosophical terms, they seem to have originated from the geometrical terminology of the Greeks.²²

²² See B. Einarson, "On Certain Mathematical Terms in Aristotle's Logic," American Journal of Philology, LVII (1936), 34-35 and 151-172, esp. 36 ff.

Traditionally, there were two main variants of the concepts of the analytic and the synthetic.²³ In one of them, a synthetic argument was an ordinary step-by-step deductive argument, whereas in an analytic argument one started from what one wanted to prove and tried to reduce it to something known from which it could be proved. The other sense of the analytic and the synthetic was tied more closely to geometry, although these ties were by no means inseparable. Forgetting certain qualifications, we may say that a geometrical argument was called analytic in the second sense in so far as no constructions were carried out in it, that is, in so far as no new lines, points, circles, and the like were introduced during the argument. An argument was called synthetic if such new entities were introduced. Here we are interested in the second sense only.

Now it is well known that if the geometrical arguments of (say) Euclid's system are "formalized" in the sense of being converted into the form of explicit logical arguments, most of them are instances of quantificational arguments. It is also fairly obvious that the distinction between the two kinds of geometrical arguments largely coincides with the distinction between the two kinds of quantificational arguments that I have been discussing. A geometrical argument in the course of which no new geometrical entities are "constructed"-that is, introduced into the discussion -will normally be converted into a quantificational argument in the course of which no new free individual symbols are introduced and the degree of the sentences in question is not otherwise increased. The geometrical notion of analyticity definable in terms of the notion of a construction will thus virtually become a special case of the sense of analyticity that we characterized by means of the notion of degree. Elsewhere, I have argued that Kant's usage of the terms "analytic" and "synthetic" largely followed the mathematical paradigm.²⁴ He made it clear, furthermore, that

²³ The history of the two versions is in many ways intertwined almost inextricably. It seems to me important to keep the two separate as much as possible, however. Some interesting remarks on the history and the interrelations of the two versions are made by Neal W. Gilbert in *The Renaissance Concepts of Method* (New York, 1959), pp. 31-35, 81-83, and 171-173. ²⁴ "Kant's Theory of Mathematics" (in Finnish), *Ajatus*, XXII (1959),

²⁴ "Kant's Theory of Mathematics" (in Finnish), Ajatus, XXII (1959), 5-85.

he had in mind only the second of the two variant senses of the analytic and the synthetic which were listed above.²⁵ If I am right, Kant's usage therefore comes pretty close to the sense in which most logical truths of quantification theory were found above to be synthetic.

There is in the historical material also a half-implicit generalization of the geometrical notion of construction which may serve to establish an even closer connection between my suggested explication of the notion of analyticity and the meaning which the term "analytic" has actually had. The part of the demonstration of a Euclidean theorem in which figures were introduced (drawn for the first time) was called the *ecthesis* or "exposition."²⁶ The same term was applied by Aristotle to a procedure, used in his syllogistic theory, that is very closely related to the rule of existential instantiation.²⁷ (Indeed, on one interpretation it virtually is this rule.) It has been suggested that Aristotle was here borrowing from Greek mathematical terminology;²⁸ but even if he was not, the two notions of ecthesis were frequently related to each other, as in fact may be done for perfectly good reasons. The result was a general but somewhat vague idea of something like the rule of existential generalization. I have suggested elsewhere that something like this idea was what Kant had in mind when he described the synthetic method of mathematics.²⁹ In fact, Kant indicates that mathematical truths are synthetic because they are based on the use of constructions. The general notion of a construction is explained by him as the introduction or "exhibition" of an individual idea (individual term, as we may equally well say) to represent a general concept, an explanation strongly reminiscent of existential instantiation.³⁰ In the light of such explanations, we may safely say that for Kant something like the

²⁵ See the *Prolegomena* (Academy Edition, IV, 276), and Kant's inaugural dissertation of the year 1770, sec.1n.

²⁶ Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, 2nd ed. (Cambridge, 1926), pp. 129-131.

²⁷ See G. Patzig, *Die Aristotelische Syllogistik* (Göttingen, 1959), pp. 166-180; J. Łukasiewicz, Aristotle's Syllogistic (Oxford, 1951), pp. 59-67.

²⁸ Einarson, op. cit., pp. 161-162.

²⁹ "Kant's Theory of Mathematics," ch. v.

³⁰ Kemp Smith (trans.), op. cit., p. 577.

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rule of existential instantiation was the paradigm of synthetic modes of reasoning in mathematics. Since we saw earlier that in a suitable system of quantification theory the rule of existential instantiation is the only one that increases the degree of the sets of sentences we are considering, this serves to relate Kant's notion of analyticity even more closely to the explication which can be given to this notion in terms of the degree of a sentence (or of a set of sentences). I think that in the light of this explication we can appreciate Kant's philosophy of mathematics and of logic much better than by means of any alternative explication of analyticity. In a sense, we may in this way even partially vindicate Kant's claim that most mathematical truths are synthetic. In certain other ways, too, we seem to have here a way of making certain parts of the traditional philosophy of mathematics more relevant to our own problems than modern philosophers sometimes think they are. What more can one ask of an explication of an old philosophical notion?³¹

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